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From the Topics of Fractal Geometry

Hiroo Fukaishi 深石博夫
Department of Mathematics, Faculty of Education
Kagawa University, Takamatsu, Kagawa 760, Japan

§1. What is a fractal?

Standard geometry has left aside as being formless to investigate fugitive patterns as the shape of a cloud, a mountain, a coastline, or a tree. Mandelbrot [5] claimed that sets far reputed exceptional should in a sense be the rule, that constructions deemed pathological should evolve naturally from very concrete problems, and that study of Nature should help old problems and yield so many new ones. He called such a new branch of mathematics fractal geometry. The word "fractal" was derived from the latin <u>fractus</u>, meaning broken.

Mandelbrot gave a definition of a fractal as a set with its Hausdorff dimension strictly greater than its topological dimension, but he pointed out that the definition is unsatisfactory as it excluds certain highly irregular sets which are thought of in the spirit of fractals.

A set is said to be self-similar provided that an arbitrary small neighborhood of each point contains a "miniture" of the whole. Not all fractal sets have self-similarity, but typical ones have that property which is useful to analyze fractals.

Note that the concept of a fractal is not topological, but it depends on a metric.

Throughout the paper a space means a complete separable metric space with a specific metric.

§2. Self-similarity

DEFINITION 1. Let X be a space with a metric d. A map $f : X \to X$ is a <u>contracation</u> if the Lipschitz constant

$$L(f) = \sup_{x \neq y} \frac{d(f(x),f(y))}{d(x,y)}$$

satisfies $L(f) < 1$. Any contraction of X has a unique fixed point.

Let 2^X be the hyperspace of X; i.e. the set of all non-empty compact subsets of X with the Hausdorff metric

$$d_H(A,B) = \max \{ \inf \{ \varepsilon > 0 : U(A;\varepsilon) \supset B \} , \inf \{ \varepsilon > 0 : U(B;\varepsilon) \supset A \} \}$$

where $A, B \in 2^X$.

THEOREM 1.([3]) Let $\{f_1, f_2, \ldots, f_m\}$ be a finite family of contractions of X. Then there is a unique compact subset K of X such that $K = f_1(K) \cup f_2(K) \cup \ldots \cup f_m(K)$.

Proof. (1) The Hausdorff metric d_H on 2^X is complete.

(2) The map $F : 2^X \to 2^X$ defined by

$$F(A) = f_1(A) \cup f_2(A) \cup \ldots \cup f_m(A) \quad \text{for each } A \in 2^X,$$

is a contraction on 2^X.

(3) For a fixed compact subset A, $\{F^n(A)\}_n$ is a Cauchy sequence in 2^X. Hence, the set $K = \lim_n F^n(A)$ is a unique fixed point of F, which satisfies the requirement in the theorem.□

DEFINITION 2. A set $K \in 2^X$ is <u>self-similar</u> if the set K can be expressed in the form

$$K = f_1(K) \cup f_2(K) \cup \ldots \cup f_m(K)$$

where $\{f_1, f_2, \ldots, f_m\}$ is a finite family of contractions of 2^X and $m \geqq 2$. The self-similar set K is called the invariant set of $\{f_1, f_2, \ldots, f_m\}$ and denoted by $K = K(f_1, f_2, \ldots, f_m)$.

THEOREM 2.([3] , [7]) Under the hypothesis in Theorem 1, the invariant set K is given by the formula:

$$K = \mathrm{Cl}(\cup \{\mathrm{Fix}(f_{i1} \circ f_{i2} \circ \ldots \circ f_{in}) : 1 \leqq i_1, i_2, \ldots, i_n \leqq m,\ n \geqq 1\}),$$

where $\mathrm{Fix}(\ )$ denotes the unique fixed point of a contaction of 2^X.

Proof. If $f_k(A) \subset A$ $(k = 1, 2, \ldots, m)$ and $f_1(A) \cup f_2(A) \cup \ldots \cup f_m(A) \supset A$ for $A \in 2^X$, then $A = f_1(A) \cup f_2(A) \cup \ldots \cup f_m(A)$.

Let $W(n)$ consist of all words of length n and let

$$f_w = f_{i1} \circ f_{i2} \circ \ldots \circ f_{in} \quad \text{for} \quad w = i_1 i_2 \ldots i_n \in W(n).$$

(1) Setting $U = \cup \{\mathrm{fix}(f_w) : w \in W(n)\}$, $U \subset K$.

(2) $K = \cup \{f_w(K) : w \in W(n)\}$ for each n.

(3) The set U is dense in K.□

§3. Hausdorff dimension of self-similar sets

DEFINITION 3. Let $\alpha > 0$ and $A \subset X$. For each $\varepsilon > 0$ let

$$\Lambda_\alpha^\varepsilon(A) = \inf\{\sum_{n=1}^{\infty} (\mathrm{diam}\, S_n)^\alpha : \{S_n\}_n \text{ is a countable cover of } A \text{ by closed spheres of diameters} < \varepsilon\}.$$

Then let

$$\Lambda_\alpha(A) = \sup_{\varepsilon > 0} \Lambda_\alpha^\varepsilon(A),$$

which is called the Hausdorff α-dimensional outer measure of A. The function Λ_α is an outer measure on X in the sense of Carathéodory.

Let $\dim_H(A) = \sup\{\alpha : \Lambda_\alpha(A) = \infty\} = \inf\{\alpha : \Lambda_\alpha(A) = 0\}$,

which is called the Hausdorff dimension of A. Mandelbrot [5] renamed it fractal dimension, because it plays a central role in fractal geometry.

THEOREM 3.([3]) Let $\{f_1, f_2, \ldots, f_m\}$ be a finite family of contractions of $X = \mathbb{R}^D$ with the Euclidean metric satisfying the following:

(i) $\| f_k(x) - f_k(y) \| = L(f_k) \| x-y \|$ for each $x, y \in X$ and for each k.

(ii) There exists a bounded open set V such that

$$f_1(V) \cup f_2(V) \cup \ldots \cup f_m(V) \subset V \quad \text{and} \quad f_i(V) \cap f_k(V) = \emptyset \quad \text{for any } i \neq k.$$

Then the Hausdorff dimension $\lambda = \dim_H(K)$ of the invariant set K is given by the formula:

$$(L(f_1))^\lambda + (L(f_2))^\lambda + \ldots + (L(f_m))^\lambda = 1.$$

EXAMPLE 1. The Cantor set C is an invariant set of two contractions $\{f_1, f_2\}$ of $X = [0,1]$ with the Euclidean metric given by

$$f_1(x) = \frac{1}{3} x \quad \text{and} \quad f_2(x) = \frac{1}{3} x + \frac{2}{3}.$$

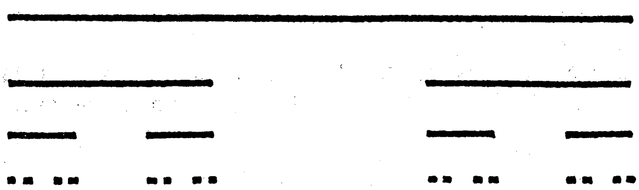

Fig.1 The Cantor set, $\dim_H(C) = (\log 2)/(\log 3) = 0.6309$

EXAMPLE 2. The Koch curve K is an invariant set of two contractions $\{f_1, f_2\}$ of $X = \mathbb{C}$ with the Euclidean metric given by

$$f_1(z) = \alpha \bar{z} \quad \text{and} \quad f_2(z) = (1-\alpha)\bar{z}+\alpha \quad \text{where} \quad \alpha = \frac{1}{2} + \frac{\sqrt{3}}{6} i.$$

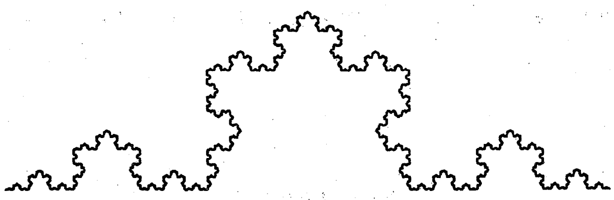

Fig.2 The Koch curve, $\dim_H(K)=(2 \log 2)/(\log 3)=1.2619$

EXAMPLE 3. Let $X = \mathbb{C}$, $f_1(z) = \alpha \bar{z}$ and $f_2(z) = |\alpha|^2+(1-|\alpha|^2)\bar{z}$ where $\alpha = \dfrac{1}{2} + \dfrac{\sqrt{3}}{6} i$. Then the invariant set $K(f_1, f_2)$ is shown as below which was found by Hata [1] and used as a cover design of a book [8] .

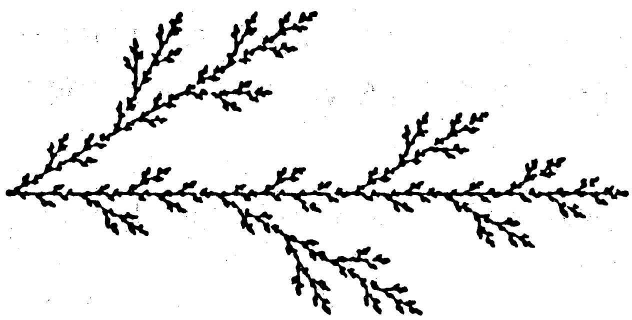

Fig.3

Hata [1] studied independently expressions of self-similar sets by means of plural contractions and extended the notion of a contraction to generalize the above theorems:

DEFINITION 4. A map $f : X \to X$ is a <u>weak contraction</u> if

$$\Omega_f(t) = \lim_{s \to t+} \omega_f(s) < t \quad \text{for any} \quad t > 0,$$

where $$\omega_f(s) = \sup\{d(f(x), f(y)) : d(x,y) \leqq s\}.$$

A contraction is a weak contraction, but not conversely. Every weak contraction has a unique fixed point in X.

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